EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

2WAB0 Final test Calculus A; 29-01-2018, 18:00–21:00

Multiple choice questions

[4] 1. The answer is C.
If $y = \arcsin(\frac{3}{5})$, then $\sin(y) = \frac{3}{5}$ and $-\pi/2 \le y \le \pi/2$.
So, $\cos(y) = \sqrt{1-(\frac{3}{5})^2} = \pm\frac{4}{5}$, and since $-\pi/2 \le y \le \pi/2$ we have $\cos(y) = \frac{4}{5}$

[4] 2. The answer is A.
We have $f(0) = 1$. The derivative is $f'(x) = 2(\sin(x) - \cos(x)) \cdot (\cos(x) + \sin(x)))$.
We have $f'(0) = -2$.
Therefore $L(x) = 1 - 2(x - 0) = 1 - 2x$.

[4] 3. The answer is D.
Implicit differentiation gives $2xy^2 + 2x^2yy' - 1 = 3y'$.
Substituting $x = 1$ and $y = 1$ gives $2 + 2y' - 1 = 3y'$. Therefore, $y' = 1$ for $x = 1$.
So the tangent line is $y = 1 \cdot (x - 1) + 1$. This gives $y = x$.

[4] 4. The answer is C.

$$\frac{6}{x-2} - \frac{3}{x-7} \le 0$$
$$\frac{6(x-7)}{(x-2)(x-7)} - \frac{3(x-2)}{(x-7)(x-2)} \le 0$$
$$\frac{6x-42-3x+6}{(x-2)(x-7)} \le 0$$

It is useful to make a sign chart

$-$	?	$+$	?	$-$	0	$+$		$\frac{3(x-12)}{(x-2)(x-7)}$
	2		7		12		$x \rightarrow$	

So we have
$$(x < 2) \vee (7 < x \le 12).$$

[4] 5. The answer is D.
Separation of variables yields:
$$\frac{y'}{y} = \frac{2}{x-2}$$
$$\frac{1}{y}\,dy = \frac{2}{x-2}\,dx$$
$$\int \frac{1}{y}\,dy = \int \frac{2}{x-2}\,dx$$
$$\ln|y| = 2\ln|x-2| + C$$
$$y = A(x-2)^2$$
Since $y(3) = 5$ we have $A = 5$. Solution: $y = 5(x-2)^2$.

Open questions

[3] 6. (a) Let $P = (0,0,6)$, $Q = (0,6,6)$ and $R = (-5,0,11)$. Then $\overrightarrow{PQ} = \langle 0,6,0\rangle$ and $\overrightarrow{PR} = \langle -5,0,5\rangle$.
The cross product of $\overrightarrow{PQ} = \langle 0,6,0\rangle$ and $\overrightarrow{PR} = \langle -5,0,5\rangle$ is equal to $\langle 30,0,30\rangle$.
This vector (or $\langle 1,0,1\rangle$) is normal to the plane (roof) and points to the sun when it shines perpendicular to the solar panels.

[2] (b) We have

$$\langle 1,0,1\rangle \cdot \langle 1,0,0\rangle = \|\langle 1,0,1\rangle\| \cdot \|\langle 1,0,0\rangle\| \cdot \cos(\alpha)$$

So $\cos(\alpha) = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$. which gives $\alpha = \frac{\pi}{4}$.
Remark: one can also use the picture to find this result.

[4] 7. Let $y = x^{x+1}$. Then $\ln(y) = (x+1)\ln(x)$ and hence

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}((x+1)\ln(x)) \iff \frac{1}{y}\cdot y' = \ln(x) + (x+1)\cdot\frac{1}{x}$$
$$\iff y' = \left(\ln(x) + \frac{x+1}{x}\right)\cdot y = \left(\ln(x) + \frac{x+1}{x}\right)\cdot x^{x+1}.$$

Alternative solution:
Write

$$f(x) = e^{\ln x^{(x+1)}} = e^{(x+1)\ln(x)}$$

and differentiate.
The derivative is

$$f'(x) = e^{(x+1)\ln(x)}\cdot\left(\ln(x) + \frac{x+1}{x}\right) = x^{x+1}\cdot\left(\ln(x) + \frac{x+1}{x}\right).$$

8. Inverse function.

[3] (a) We have $y = \dfrac{3}{\ln(x)-1}$, so $y(\ln(x)-1) = 3$. Therefore $\ln(x) = \dfrac{3}{y} + 1$, and $x = e^{\frac{y+3}{y}}$.
The inverse function is $f^{-1}(y) = e^{\frac{y+3}{y}}$

[2] (b) The range of the inverse function is the domain of $f(x)$. This domain is given and is (e,∞).
The domain of the inverse function is the range of $f(x)$. The derivative of $f(x)$ is $\frac{3}{(\ln(x)-1)^2}\cdot\frac{-1}{x}$.
We have $f'(x) < 0$, so the function is decreasing.
Since $\lim_{x\to e^+} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = 0$, the range of $f(x)$ is $(0,\infty)$.

9. Box

(a) The area of the bottom is x^2. The costs for the bottom are $2ax^2$. The costs for the top are ax^2. The total area of the four sides is $4xy$. The costs for the sides are $4axy$. The total costs are
[2] $(3x^2 + 4xy)a$.

[3] (b) We have $V = 1000$, so $yx^2 = 1000$ and $y = 1000/x^2$. Substitution in the formula for the total costs gives $(3x^2 + \frac{4V}{x})a$. To find the minimum value we take the derivative with respect to x. The derivative is $(6x - \frac{4V}{x^2})a$. This is equal to zero for $6x^3 = 4\cdot 1000$, so $x = \sqrt[3]{\frac{2}{3}}\cdot 10$ and indeed this gives a minimum. For y we find $y = \frac{1000}{x^2} = \frac{1000}{\sqrt[3]{\frac{4}{9}}\cdot 100} = \sqrt[3]{\frac{9}{4}}\cdot 10$.

[2] 10. Evaluate

$$\lim_{x\to 0}\frac{\ln(1+x)-x}{1-\cos x}.$$

Since this has the indeterminate form $0/0$, one can apply l'Hôpital.

$$\lim_{x\to 0}\frac{\ln(1+x)-x}{1-\cos x} = \lim_{x\to 0}\frac{(\ln(1+x)-x)'}{(1-\cos x)'} = \lim_{x\to 0}\frac{\frac{1}{1+x}-1}{\sin x}.$$

At this point one can evaluate the limit directly or apply l'Hôpital once more:

$$\lim_{x\to 0} \frac{\frac{1}{1+x} - 1}{\sin x} = \left\{ \begin{array}{l} \lim_{x\to 0} \frac{x}{\sin x} \cdot \frac{-1}{1+x} \\ \text{or} \\ \lim_{x\to 0} \frac{(\frac{1}{1+x}-1)'}{(\sin x)'} = \lim_{x\to 0} \frac{\frac{-1}{(1+x)^2}}{\cos x} \end{array} \right\} = -1 \ .$$

[4] 11. To find this integral, use the substitution $u = 1 + x^2$.
This gives $du = 2x\,dx$, and we find

$$\int_0^{2\sqrt{2}} x^3 \sqrt{x^2+1}\,dx =$$

$$\int_0^{2\sqrt{2}} \tfrac{1}{2}x^2 \cdot 2x \sqrt{x^2+1}\,dx =$$

$$\int_1^9 \tfrac{1}{2}(u-1)\sqrt{u}\,du =$$

$$\int_1^9 \left(\tfrac{1}{2}u^{3/2} - \tfrac{1}{2}u^{1/2}\right)\,du =$$

$$\left(\tfrac{1}{5}u^{5/2} - \tfrac{1}{3}u^{3/2}\right)\Big|_1^9 =$$

$$\tfrac{1}{5}3^5 - \tfrac{1}{3}3^3 - (\tfrac{1}{5} - \tfrac{1}{3}) =$$

$$\tfrac{243}{5} - 9 - \tfrac{1}{5} + \tfrac{1}{3} = \tfrac{242}{5} - \tfrac{26}{3}.$$

[3] 12. Let $I = \int e^{2x}\,\sin(x)\,dx$. Use integration by parts with $u = e^{2x}$ and $dv = \sin(x)\,dx$.
This choice gives $du = 2e^{2x}\,dx$ and $v = -\cos(x)$.
We find

$$I = -e^{2x}\,\cos(x) + \int 2e^{2x}\,\cos(x)\,dx$$

Use again integration by parts with $u = 2e^{2x}$ and $dv = \cos(x)\,dx$.
This choice gives $du = 4e^{2x}\,dx$ and $v = \sin(x)$.
We find

$$I = -e^{2x}\,\cos(x) + 2e^{2x}\,\sin(x) - 4\int e^{2x}\,\sin(x)\,dx$$

From this we find

$$5I = -e^{2x}\,\cos(x) + 2e^{2x}\,\sin(x) + D,$$

so

$$I = -\tfrac{1}{5}e^{2x}\,\cos(x) + \tfrac{2}{5}e^{2x}\,\sin(x) + C.$$