EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

2WAB1 Final test Calculus A; 2-11-2016, 9:00–12:00

Multiple choice questions

[4] 1. The answer is C, the integral diverges, because

$$\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{R\to\infty}\int_0^R \frac{x}{1+x^2}\,dx = \lim_{R\to\infty}\left[\tfrac{1}{2}\ln(1+x^2)\right]_0^R = \lim_{R\to\infty}\tfrac{1}{2}\ln(1+R^2) = \infty$$

[4] 2. $f'(x) = -2xe^{-x^2}$, so $f'(x) = 0 \iff x = 0$. Make a sign chart of f':

$f'(x)$		$+$	0	$-$	
x	-3		0		2

Function has a local maximum $f(0) = 1$ at $x = 0$. At the endpoints of the interval the function attains the values $f(-3) = e^{-9}$ for $x = -3$ and $f(2) = e^{-4}$ for $x = 2$. The answer is D, the maximum is 1 and the minimum is e^{-9}

[4] 3. The answer is A, since $\displaystyle\lim_{x\to 0}\frac{x^5+3x^3+2x}{4x^5+4x^4+x} = \lim_{x\to 0}\frac{x^4+3x^2+2}{4x^4+4x^3+1} = 2$

[4] 4. The answer is C, since $\arcsin(\sin(\tfrac{3}{4}\pi)) = \arcsin(\tfrac{1}{2}\sqrt{2}) = \tfrac{1}{4}\pi$

[4] 5. Implicit differentiation gives answer A:

$$2xy^2 + 2x^2yy' - 2 = -4y' \iff (2x^2y+4)y' = 2 - 2xy^2 \iff y' = \frac{1-xy^2}{2+x^2y}$$

Open questions

[4] 6. We have $\dfrac{e^x+e^{-x}}{2}+3\dfrac{e^x-e^{-x}}{2}+1=0 \quad \Longleftrightarrow 4e^x-2e^{-x}+2=0.$

Let $z:=e^x$. Then $4z^2+2z-2=0$, so $2z^2+z-1=0$, and we find $z=\frac{-1\pm\sqrt{1+8}}{4}=\frac{-1\pm 3}{4}$

Since z needs to be positive, we reject $z=-1$

The other solution, $z=\frac{1}{2}$, leads to $x=\ln(\frac{1}{2})$

[4] 7. The cross product of the two direction vectors gives a normal vector for the plane: $\mathbf{n}=\begin{pmatrix}1\\2\\2\end{pmatrix}\times$

$\begin{pmatrix}1\\-3\\1\end{pmatrix}=\begin{pmatrix}8\\1\\-5\end{pmatrix} \qquad \Longrightarrow$ Plane has equation $8x+y-5z=C$

By substituting e.g. the point $(1,2,3)$ into this equation we find $C=-5$

Alternative approach: write directly $8(x-1)+(y-2)-5(z-3)=0$. Of course: one can choose another point than $(1,2,3)$. If a wrong point is chosen (for example $(1,2,2)$): only 1 point.

[2] 8. (a) $f'(x)=3x^2+6x$, so $f'(1)=9$

The tangent line is given by $y=f'(1)(x-1)+f(1)=9(x-1)+3=9x-6$

[2] (b) $9x-6=0 \iff x=\frac{2}{3}$

[2] 9. (a) $f'(x)=\dfrac{2\sqrt{1+x^2}-2x\frac{1}{2\sqrt{1+x^2}}2x}{1+x^2}=\dfrac{2(1+x^2)-2x^2}{(1+x^2)^{\frac{3}{2}}}=\dfrac{2}{(1+x^2)^{\frac{3}{2}}}>0,$

so the function is strictly increasing

[1] (b) The range of f^{-1} is equal to the domain of f. The function f is defined for all $x\in\mathbb{R}$, so the range of f^{-1} is $\mathbb{R}$

[3] 10. First use the substitution $w=\sin(x)$. Then $dw=\cos(x)\,dx$:

$$I:=\int \ln(\sin(x))\cdot\cos(x)\,dx=\int \ln(w)\,dw$$

Next apply integration by parts: $u=\ln(w)$, so $du=\frac{1}{w}\,dw$, and $dv=dw$, so $v=w$:

$$\begin{aligned}I&=uv-\int v\,du=w\ln(w)-\int w\frac{1}{w}\,dw\\&=w\ln(w)-w+C=\sin(x)\ln(\sin(x))-\sin(x)+C\end{aligned}$$

Alternative approach: Use integration by parts immediately with $u=\ln(\sin(x))$ and $dv=\cos(x)\,dx$

$$\begin{aligned}I&=uv-\int v\,du=\sin(x)\ln(\sin(x))-\int\frac{\cos(x)}{\sin(x)}\sin(x)\,dx\\&=\sin(x)\ln(\sin(x))-\int\cos(x)\,dx=\sin(x)\ln(\sin(x))-\sin(x)+C\end{aligned}$$

[2] 11. (a) $A = 2\pi r^2 + 2\pi r h$

[2] (b) Since $V = \pi r^2 h = 330$, we find $h = \frac{330}{\pi r^2}$. Express A in r only: $A = 2\pi r^2 + \frac{660}{r}$

Differentiate, $\frac{dA}{dr} = 4\pi r - \frac{660}{r^2}$, and set derivative to zero to find the location of the minimum

$$\frac{dA}{dr} = 0 \iff 4\pi r = \frac{660}{r^2} \iff r^3 = \frac{165}{\pi} \iff r = \sqrt[3]{\frac{165}{\pi}}$$

The corresponding value for h can be found with $h = \frac{330}{\pi r^2}$

[4] 12. $f(x) = x^3$ so $f'(x) = 3x^2$ and we get

$$A = 2\pi \int_0^1 f(x)\sqrt{1 + (f'(x))^2}\,dx = 2\pi \int_0^1 x^3\sqrt{1 + 9x^4}\,dx$$

Use the substitution $u = 1 + 9x^4$. Then $du = 36x^3\,dx$. The lower limit $x = 0$ gives $u = 1$, the upper limit $x = 1$ gives $u = 10$, so we have

$$A = 2\pi \int_1^{10} \tfrac{1}{36}\sqrt{u}\,du \quad = \frac{\pi}{18}\left[\tfrac{2}{3}u^{\frac{3}{2}}\right]_1^{10} = \frac{\pi}{27}\left(10\sqrt{10} - 1\right)$$

[2] 13. (a) $y' = ky \iff \dfrac{dy}{dt} = ky \iff \dfrac{1}{y}\,dy = k\,dt \iff \ln(|y|) = kt + C$

Because $y > 0$ we have $y = e^{kt+C} = e^{kt}e^C = Ae^{kt}$. The constant A follows from the initial condition $y(0) = 88$ and we get $A = 88$ and $y = 88e^{kt}$

If we also use $y(12.3) = 44$ then we find $88e^{12.3k} = 44 \iff e^{12.3k} = \frac{1}{2} \iff 12.3k = \ln(\frac{1}{2}) = -\ln(2) \iff k = -\ln(2)/12.3$

[2] (b) 88 gram $\longrightarrow$ 44 gram $\longrightarrow$ 22 gram $\longrightarrow$ 11 gram

It takes three half-life periods, so $3 \cdot 12.3 = 36.9$ years