Multiple choice questions

1. Multiple Choice Questions

   (a) **English**: The complete solution of the inequality $\dfrac{8}{x-1} \leq \dfrac{2}{x-7}$ is given by

   Dutch: De volledige oplossing van de ongelijkheid $\dfrac{8}{x-1} \leq \dfrac{2}{x-7}$ wordt gegeven door

   i. $(1 < x < 7)$
   ii. $(x \leq 1) \vee (7 \leq x)$
   iii. $(x < 1) \vee (7 < x)$
   iv. $(x < 1) \vee (7 < x \leq 9)$

   Solution: (iv) The inequality $\frac{8}{x-1} \leq \frac{2}{x-7}$ gives $\frac{8}{x-1} - \frac{2}{x-7} \leq 0$.
   Reducing to the same denominator gives

$$\frac{8(x-7)}{(x-1)(x-7)} - \frac{2(x-1)}{(x-7)(x-1)} \leq 0.$$

   This gives $\dfrac{8x-56-2x+2}{(x-1)(x-7)} \leq 0$ and $\dfrac{6(x-9)}{(x-1)(x-7)} \leq 0$.
   A picture with the sign of the function on the left-hand side of the inequality gives

$-$	?	$+$	?	$-$	0	$+$	$\frac{(x-9)}{(x-1)(x-7)}$
	1		7		9		$x \rightarrow$

   So the final result is

$$(x < 1) \vee (7 < x \leq 9).$$ **End solution**.

(b) **English**: Which line passes through the points $(2,4)$ and $(3,1)$?
Dutch: Welke lijn gaat door de punten $(2,4)$ en $(3,1)$?

i. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

ii. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

iii. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

iv. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Solution: (iii). A direction vector is $\langle 3-2, 1-4\rangle = \langle 1, -3\rangle$. Therefore the answer is (c) or (d). We check if the point $(3,1)$ is on the line. For $\lambda = 1$ we find indeed that the point is on the line given in (c). For (d) it is not possible to choose a λ such that $(3,1)$ is on the line. **End solution**.

(c) **English**: Consider the following function $f(x) = \begin{cases} x^2 + 3x - 3 & \text{if } x < 1, \\ 5x - 4 & \text{if } x \geq 1. \end{cases}$

In $x = 1$ the function is

Dutch: Beschouw de volgende functie $f(x) = \begin{cases} x^2 + 3x - 3 & \text{if } x < 1, \\ 5x - 4 & \text{if } x \geq 1. \end{cases}$

In $x = 1$ is de functie

i. continuous and differentiable
continu en differentieerbaar

ii. not continuous but differentiable
niet continu maar wel differentieerbaar

iii. continuous but not differentiable
continu maar niet differentieerbaar

iv. not continuous and not differentiable
niet continu en niet differentieerbaar

Solution:(i). We have $\lim_{x\to 1^-} f(x) = 1 + 3 - 3 = 1$ and $\lim_{x\to 1^+} f(x) = 5 - 4 = 1$. So the function is continuous. The derivative of the function is for $x < 1$ equal to $2x + 3$ and for $x > 1$ equal to 5. The limit of the derivative is for $x \to 1^+$ and for $x \to 1^-$ the same. Therefore the function is also differentiable. **End solution**.

(d) **English**: Find the absolute (or global) extrema of the given function on the indicated interval.
Dutch: Geef de absolute (of globale) extrema van de gegeven functie op het gegeven interval.

$$f(x) = \frac{x}{x^2+16} \text{ on } [-2,6].$$

i. absolute minimum $-\frac{1}{10}$, absolute maximum $\frac{1}{4}$
ii. absolute minimum $\frac{1}{10}$, absolute maximum $\frac{3}{26}$
iii. absolute minimum $-\frac{1}{10}$, absolute maximum $\frac{1}{8}$
iv. absolute minimum 0, absolute maximum $\frac{1}{4}$

Solution:(iii). We have $f'(x) = \frac{x^2+16-2x^2}{(x^2+16)^2} = \frac{16-x^2}{(x^2+16)^2}$. The derivative is positive for $-2 < x < 4$ and negative for $4 < x < 6$. Therefore we have a global maximum $1/8$ in $x = 4$. Candidates for the absolute minimum are $f(-2) = -1/10$ and $f(6) = 3/26$. So the absolute minimum is $-1/10$. **End solution**.

(e) **English**: The slope of the tangent line at point $(-3,1)$ of curve $x^2y + xy^2 = 6$ is
Dutch: De richtingscoëfficiënt van de raaklijn in het punt $(-3,1)$ aan de kromme $x^2y + xy^2 = 6$ is

i. 1
ii. $-\frac{3}{5}$
iii. $\frac{5}{3}$
iv. -2

Solution:(iii). Implicit differentiation gives $2xy + x^2y' + y^2 + 2xyy' = 0$.
Substituting $x = -3$ and $y = 1$ gives $-6 + 9y'(-3) + 1 - 6y'(-3) = 0$. So, $3y'(-3) = 5$ and $y'(-3) = 5/3$. **End solution**.

(f) **English**: Determine $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x^2+x}-x}$.

Dutch: Bepaal $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x^2+x}-x}$.

i. 0
ii. 2
iii. $\frac{1}{2}$
iv. 1

Solution:(ii). Multiply by $\dfrac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$.
We find $\dfrac{1}{\sqrt{x^2+x}-x} \cdot \dfrac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \dfrac{\sqrt{x^2+x}+x}{x}$.
Now we find $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x^2+x}-x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+x}+x}{x}$.
To find the limit we divide numerator and denominator by x to obtain
$\lim\limits_{x\to\infty} \sqrt{1+\dfrac{1}{x}}+1 = 2$. **End solution**.

(g) **English**: Determine $\tan(\arccos(-\frac{12}{13}))$.
Dutch: Bepaal $\tan(\arccos(-\frac{12}{13}))$.

i. $\dfrac{169}{144}$
ii. $\dfrac{5}{13}$
iii. $-\dfrac{5}{13}$
iv. $-\dfrac{5}{12}$

Solution:(iv). Define $y = \arccos(-\frac{12}{13})$.
Then by definition $\cos(y) = -\frac{12}{13}$ and $0 \le y \le \pi$.
So, $\sin^2(y) = 1-(-\frac{12}{13})^2 = \frac{25}{169}$. Since $0 \le y \le \pi$, we find $\sin(y) = \frac{5}{13}$. Therefore, $\tan(y) = \frac{\sin(y)}{\cos(y)} = \frac{5/13}{-12/13} = -\frac{5}{12}$. **End solution**.

(h) **English**: Determine $\int \frac{x}{\sqrt{1+x^2}}\,dx$.

Dutch: Bepaal $\int \frac{x}{\sqrt{1+x^2}}\,dx$.

i. $\sqrt{1+x^2}+C$

ii. $\frac{1}{\sqrt{1+x^2}}+C$

iii. $\frac{1}{2}\left(\arctan(x)-x\sqrt{1+x^2}\right)+C$

iv. $\ln|x+1|+\frac{1}{\sqrt{1+x^2}}+C$

Solution:(i). Use the substitution $u=x^2+1$. Then we have $du=2x\,dx$. This gives

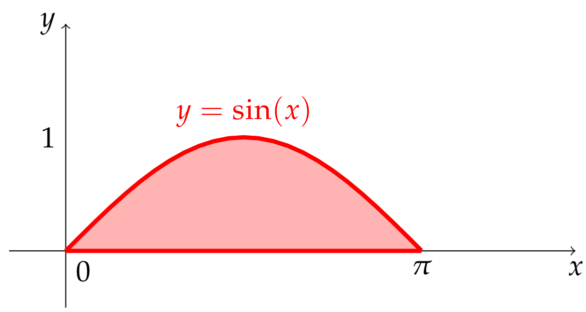
$$\int \frac{x}{\sqrt{1+x^2}}\,dx=\int \frac{1}{2\sqrt{u}}\,du=\sqrt{u}+C=\sqrt{1+x^2}+C.$$

End solution.

Open questions

2. Consider the graph of the function $f(x)=\sin(x)$ on the interval $[0,\pi]$. In this question we compute the center of gravity of the region enclosed by the graph and the x-axis. See the figure below.

Beschouw de grafiek van de functie $f(x)=\sin(x)$ op het interval $[0,\pi]$. In deze vraag berekenen we het zwaartepunt van het gebied dat wordt omsloten door de grafiek en de x-as. Zie de figuur hieronder.



[1] (a) We denote the area of the red region by A. Compute A.
We noemen de oppervlakte van het rode gebied A. Bereken A.

[2] (b) The x-coordinate of the center of gravity is given by $x_C=\frac{1}{A}\int_0^{\pi} x\sin(x)\,dx$.
Compute x_C.
De x-coördinaat van het zwaartepunt is $x_C=\frac{1}{A}\int_0^{\pi} x\sin(x)\,dx$.
Bereken x_C.

[2] (c) The y-coordinate of the center of gravity is given by $y_C=\frac{1}{2A}\int_0^{\pi} \sin^2(x)\,dx$.
Compute y_C.
De y-coördinaat van het zwaartepunt is $y_C=\frac{1}{2A}\int_0^{\pi} \sin^2(x)\,dx$.
Bereken y_C.

Solution

(a)

$$A = \int_0^\pi \sin(x)\,dx = -\cos(x)\Big|_{x=0}^{x=\pi} = 1 + 1 = 2$$

(b)

$$\begin{aligned} x_C &= \frac{1}{2}\int_0^\pi x\sin(x)\,dx = \frac{1}{2}\int_0^\pi x\,d(-\cos(x)) \\ &= -\frac{1}{2}x\cos(x)\Big|_{x=0}^{x=\pi} + \frac{1}{2}\int_0^\pi \cos(x)\,dx = \frac{\pi}{2} \end{aligned}$$

Alternative: Symmetry argument

(c)

$$\begin{aligned} y_C &= \frac{1}{4}\int_0^\pi \sin^2(x)\,dx = \frac{1}{4}\int_0^\pi \frac{1}{2}(1-\cos(2x))\,dx \\ &= \frac{1}{8}(x - \frac{1}{2}\sin(2x))\Big|_{x=0}^{x=\pi} = \frac{\pi}{8} \end{aligned}$$

[3] 3. Consider the planes $x + z = 3$ and $x - 4y + 3z = 5$.

Find a vector equation for the line of intersection of these planes.

Gegeven zijn de vlakken $x + z = 3$ en $x - 4y + 3z = 5$.

Bepaal een vectorvoorstelling van de snijlijn van deze vlakken.

Solution

Let $x = t$. Then $z = 3 - x = 3 - t$.

Also: $4y = x + 3z - 5$, so $y = \frac{1}{4}x + \frac{3}{4}z - \frac{5}{4} = -\frac{1}{2}t + 1$.

This gives: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -\frac{1}{2} \\ -1 \end{pmatrix}$

[4] 4. A room is 6 m high and 10 m wide. We want to hang a cable across the width of the room for attaching LED-lights (see figure below).

In the middle the cable should be 3 meters above the floor. The shape of the hanging cable is the catenary given by

$$y = f(x) = a\cosh\frac{x}{b},$$

with $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

Find a and b.

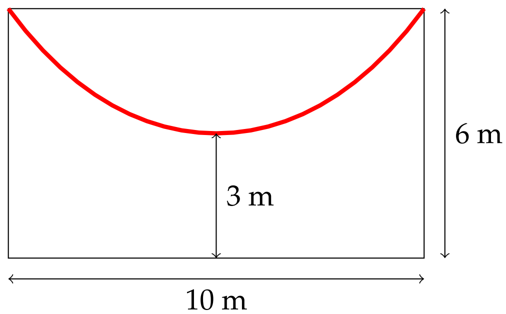
Een zaal is 6 meter hoog en 10 meter breed. Men wil over de hele breedte een ketting hangen om daar LED-lampjes aan te kunnen bevestigen (zie figuur).

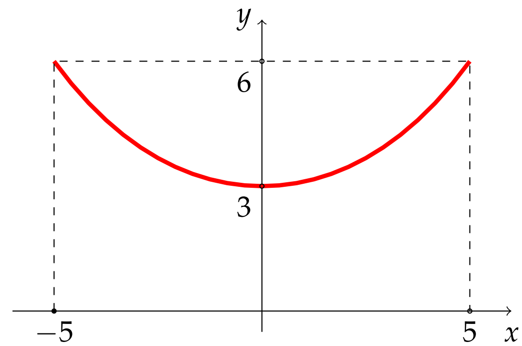
In het midden moet de ketting 3 meter boven de vloer zijn. De kettinglijn kan worden beschreven met de vergelijking

$$y = f(x) = a\cosh\frac{x}{b},$$

met $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

Bepaal a en b.





Solution

We know that $f(0) = a\cosh\frac{0}{b} = a = 3$. Therefore $a = 3$.
Furthermore we have $f(5) = 6$. This gives

$$3\cosh\frac{5}{b} = 6 \iff \cosh\frac{5}{b} = 2 \iff \cosh z = 2$$

if we let $z = 5/b$. This leads to $e^z + e^{-z} = 4$. Define $u := e^z$. Then $u + \frac{1}{u} = 4$, hence $u^2 - 4u + 1 = 0$.
This quadratic equation in u has two solutions: $u_1 = 2+\sqrt{3}$ and $u_2 = 2-\sqrt{3}$. The corresponding values for z are $z_1 = \ln(2+\sqrt{3})$ and $z_2 = \ln(2-\sqrt{3})$. Since $b = \frac{5}{z}$, we finally have $b_1 = 5/\ln(2+\sqrt{3})$ and $b_2 = 5/\ln(2-\sqrt{3})$. Note that

$$b_2 = \frac{5}{\ln(2-\sqrt{3})} = \frac{5}{\ln((2-\sqrt{3})\frac{2+\sqrt{3}}{2+\sqrt{3}})} = \frac{5}{\ln(\frac{4-3}{2+\sqrt{3}})} = \frac{5}{\ln((2+\sqrt{3})^{-1})} = -b_1.$$

Either value for b gives the same catenary.

[4] 5. Evaluate

Bepaal

$$\lim_{x\to 0}\frac{x-\sin(x)}{e^{2x}-1-2x-2x^2}.$$

Solution

The limit is of the form $0/0$.

Use l'Hôpital:

$$\lim_{x\to 0}\frac{x-\sin(x)}{e^{2x}-1-2x-2x^2} = \lim_{x\to 0}\frac{1-\cos(x)}{2e^{2x}-2-4x} =: L$$

Still $0/0$. Use l'Hôpital again

$$L = \lim_{x\to 0}\frac{\sin(x)}{4e^{2x}-4}$$

Still $0/0$. Use l'Hôpital again

$$L = \lim_{x\to 0}\frac{\cos(x)}{8e^{2x}} = \frac{1}{8}$$

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + 2x^2 + 7x - 3$.
   Beschouw de functie $f : \mathbb{R} \to \mathbb{R}$ gegeven door $f(x) = x^3 + 2x^2 + 7x - 3$.

[2] (a) Show that f has an inverse.
Laat zien dat f een inverse heeft.

[2] (b) Find $(f^{-1})'(7)$.
Bepaal $(f^{-1})'(7)$.

Solution

(a) $f'(x) = 3x^2 + 4x + 7$ has no zeros in $\mathbb{R}$. Hence $f'(x) > 0$ for all $x \in \mathbb{R}$.
The function f is increasing, so it has an inverse

(b) Use

$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))}.$$

Because $f(1) = 1 + 2 + 7 - 3 = 7$, it follows that $f^{-1}(7) = 1$. Finally: $(f^{-1})'(7) = 1/f'(1) = 1/(3 + 4 + 7) = 1/14$

[3] 7. Consider the function $f(x) = \frac{2}{3}(x-1)^{\frac{3}{2}}$.
Find the arc length of the graph $y = f(x)$ between $x = 9$ and $x = 36$.
Beschouw de functie $f(x) = \frac{2}{3}(x-1)^{\frac{3}{2}}$.
Bepaal de booglengte van de grafiek $y = f(x)$ tussen $x = 9$ en $x = 36$.

Solution

$f'(x) = \frac{2}{3}\frac{3}{2}(x-1)^{\frac{1}{2}} = (x-1)^{\frac{1}{2}}$

Hence $\sqrt{1 + (f'(x))^2} = \sqrt{1 + x - 1} = \sqrt{x}$, and

$$\begin{aligned} L &= \int_9^{36} \sqrt{1 + (f'(x))^2}\, dx = \int_9^{36} \sqrt{x}\, dx \\ &= \frac{2}{3} x^{\frac{3}{2}} \Big|_{x=9}^{x=36} = \frac{2}{3}(36 \cdot 6 - 9 \cdot 3) = 126 \end{aligned}$$

[3] 8. Let y be a function of t. Solve the initial value problem
Laat y een functie van t zijn. Los het volgende beginwaardeprobleem op

$$\begin{cases} y' = \dfrac{2y}{t-1}, \\ y(2) = 5. \end{cases}$$

Solution

Use separation of variables:

$$\begin{aligned} (t-1)\, dy = 2y\, dt \iff& \frac{1}{y}\, dy = \frac{2}{t-1}\, dt \\ \iff& \ln(|y|) = 2\ln(|t-1|) + C \\ \iff& \ln(|y|) = \ln((t-1)^2) + C \iff |y| = A(t-1)^2 \end{aligned}$$

Initial condition: $y(2) = 5$, so $A = 5$. Conclusion: $y = 5(t-1)^2$

9. An internet warehouse wants to reduce its packaging costs. Given is a square sheet of cardboard with dimensions 30 cm by 30 cm. We cut off four little squares at the four corners of the sheet and fold along the dashed lines to create a box. Note that we get an open box (it has no top) in this way. See Figure 1.

   Our goal is to maximize the volume V of the open box.

   Een internetwinkel wil de verpakkingskosten beperken. Gegeven is een vierkant stuk karton met afmeting 30 cm bij 30 cm. We snijden vier kleine vierkantjes af bij de vier hoeken van het stuk karton en vouwen het karton langs de stippellijnen om een doos te maken. Merk op dat we op deze manier een open doos krijgen. Zie Figuur 1.

   Het doel is om het volume V van de doos te maximaliseren.

[2] (a) Show that $V = 4x^3 - 120x^2 + 900x$.
Laat zien dat $V = 4x^3 - 120x^2 + 900x$.

[2] (b) Find the length x of the little squares such that V has maximum value.
Bepaal de lengte x van de zijde van de kleine vierkantjes zo dat V maximaal is.

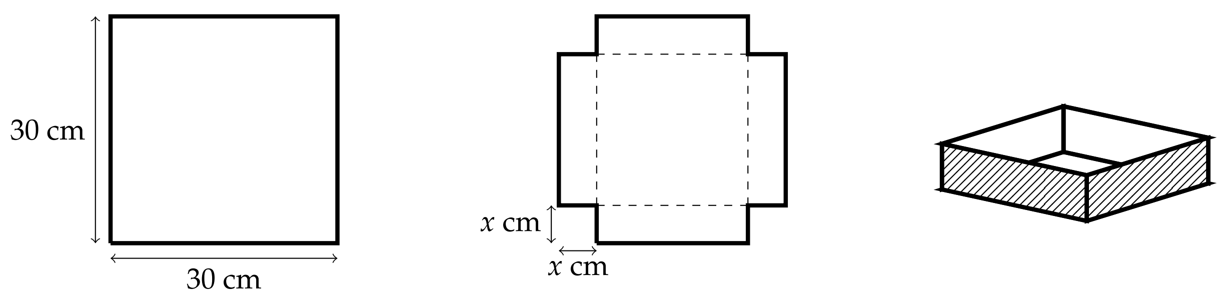


Figure 1: Making an open box from a square sheet of cardboard. Het maken van een open doos uit een vierkant stuk karton.

Solution

(a) Area of the bottom of the box is $(30-2x)^2$
Height is x
So $V = x(30-2x)^2 = x(900-120x+4x^2) = 4x^3 - 120x^2 + 900x$

(b) $V' = 12x^2 - 240x + 900 = 12(x^2 - 20x + 75) = 12(x-15)(x-5)$, so $V' = 0$ for $x = 5$ and $x = 15$
Sign chart:

	$+$	0	$-$	0	$+$	$V'(x)$
		5		15		x

The volume is maximized for $x = 5$