EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

2WAB0 Final test Calculus A; 1-11-2017, 9:00–12:00

Multiple choice questions

[4] 1. The answer is B, since $\lim\limits_{x\to 0^+} \dfrac{210x^{-0.7}+80}{7x^{-0.7}+8} = \lim\limits_{x\to 0^+} \dfrac{210+80x^{0.7}}{7+8x^{0.7}} = \dfrac{210}{7} = 30$

[4] 2. The answer is B. The derivative is

$$f'(x) = \frac{x^2+25-x\cdot 2x}{(x^2+25)^2} = \frac{-x^2+25}{(x^2+25)^2},$$

so $f'(x) = 0$ for $x = -5$ and $x = 5$. Make a sign chart:

$$\begin{array}{ccccc|c} - & 0 & + & 0 & - & f'(x) \\ \hline & -5 & & 5 & & x \end{array}$$

Since the domain is $[-1, 6]$, extrema may occur at $x = -1$, $x = 5$ and $x = 6$. The function values are $f(-1) = -\frac{1}{26}$, $f(5) = \frac{1}{10}$ and $f(6) = \frac{6}{61}$. We conclude that the global maximum is $\frac{1}{10}$ and the global minimum is $-\frac{1}{26}$

[4] 3. The answer is D. If we set $G(x) = \int_0^x (t^2+2)\,dt$, then $G'(x) = x^2+2$ (FTC2). Here we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_{2x}^{\cos(x)} (t^2+2)\,dt = \frac{d}{dx}\left(\int_{2x}^{0}(t^2+2)\,dt + \int_0^{\cos(x)}(t^2+2)\,dt\right) \\ &= \frac{d}{dx}\left(-\int_0^{2x}(t^2+2)\,dt + \int_0^{\cos(x)}(t^2+2)\,dt\right) = \frac{d}{dx}\left(-G(2x)+G(\cos(x))\right) \\ &= -G'(2x)\cdot 2 + G'(\cos(x))\cdot(-\sin(x)) = -8x^2-4-\cos^2(x)\sin(x)-2\sin(x) \end{aligned}$$

[4] 4. The answer is A. Implicit differentiation gives:

$$\begin{aligned} \cos(x) &= e^{-y\cos(x)}\cdot\left(-\frac{dy}{dx}\cos(x) - y\cdot(-\sin(x))\right) \\ \cos(x)e^{y\cos(x)} &= -\frac{dy}{dx}\cos(x) + y\sin(x) \\ \frac{dy}{dx} &= y\tan(x) - e^{y\cos(x)} \end{aligned}$$

[4] 5. The answer is D. We have

$$\frac{dy}{dt} = 5t^3y^2 \iff \frac{1}{y^2}\,dy = 5t^3\,dt \iff \int \frac{1}{y^2}\,dy = \int 5t^3\,dt \iff -\frac{1}{y} = \tfrac{5}{4}t^4 + C$$

$$\iff \frac{1}{y} = -\tfrac{5}{4}t^4 + \widetilde{C} \iff y = \frac{1}{-\frac{5}{4}t^4 + \widetilde{C}} \iff y = -\frac{4}{5t^4 + \widehat{C}}$$

Open questions

[2] 6. (a) For l, we have $x = -s$, $y = 3 + s$ and $z = 3$. For m, we have $x = 5 + 2t$, $y = 8 + 3t$ and $z = 1 - t$. At the intersection point, we find

$$-s = 5 + 2t, \quad 3 + s = 8 + 3t, \quad 3 = 1 - t.$$

These three equations are true for $s = -1$ and $t = -2$

[1] (b) Find a vector that is orthogonal to the direction vectors of l and m, e.g. the cross product of the direction vectors. This gives

$$\mathbf{n} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot (-1) - 0 \cdot 3 \\ 0 \cdot 2 - (-1) \cdot (-1) \\ (-1) \cdot 3 - 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -5 \end{pmatrix}$$

[2] (c) General equation: $ax + by + cz = d$. With the normal vector found in the previous question: $-x - y - 5z = d$. Now substitute a point that lies in the plane to find the value of d. Let us use the point $\langle 0, 3, 3\rangle$ that lies on l and hence also lies in the plane: $d = -0 - 3 - 5 \cdot 3 = -18$. Conclusion: $-x - y - 5z = -18$ or $x + y + 5z = 18$

Alternative solution: The plane can also be given by a vector equation, e.g., $\langle x, y, z\rangle = \langle 0, 3, 3\rangle + s\langle -1, 1, 0\rangle + t\langle 2, 3, -1\rangle$

[4] 7. Replace $\cosh(x)$ and $\sinh(x)$ by their definitions to find

$$-7 \cdot \frac{e^x + e^{-x}}{2} + 8 \cdot \frac{e^x - e^{-x}}{2} = 1$$

Rewrite as

$$-7(e^x + e^{-x}) + 8(e^x - e^{-x}) = 2 \iff e^x - 15e^{-x} = 2$$

and set $z = e^x$ to find

$$z - \frac{15}{z} = 2 \iff z^2 - 15 = 2z \iff z^2 - 2z - 15 = 0 \iff (z-5)(z+3) = 0$$

Conclusion: $z = 5$ or $z = -3$. Because $z = e^x > 0$, only $z = 5$ is a solution and we have $e^x = 5$, so $x = \ln(5)$.

[2] 8. (a) $f'(x) = \dfrac{1}{2}(x^5 + x + 2)^{-\frac{1}{2}} \cdot (5x^4 + 1) = \dfrac{5x^4 + 1}{2\sqrt{x^5 + x + 2}}$

Because $f'(x) > 0$ for $x > -1$, the function f is increasing and hence one-to-one. This means that f has an inverse

[3] (b) Since f and g are each other's inverse, we have $g'(6) = \dfrac{1}{f'(g(6))}$. By trial and error, we find $f(2) = \sqrt{2^5 + 2 + 2} = \sqrt{32 + 4} = \sqrt{36} = 6$, so $g(6) = 2$. Conclusion: $g'(6) = \dfrac{1}{f'(2)} = \dfrac{4}{27}$

[4] 9. Let $y = x^x$. Then $\ln(y) = x\ln(x)$ and hence

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x\ln(x)) \iff \frac{1}{y}\cdot y' = \ln(x) + x\cdot\frac{1}{x}$$
$$\iff y' = (\ln(x)+1)\cdot y = (\ln(x)+1)\cdot x^x$$

Alternative solution: Write $f(x) = e^{\ln(x^x)} = e^{x\ln(x)}$ and differentiate

[4] 10. We have: $f'(x) = x^{\frac{2}{3}}$, and hence $1 + (f'(x))^2 = 1 + x^{\frac{4}{3}}$. Therefore

$$A = 2\pi\int_0^1 f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi\int_0^1 \tfrac{3}{5}x^{\frac{5}{3}}\sqrt{1+x^{\frac{4}{3}}}\,dx = \tfrac{6\pi}{5}\int_0^1 x^{\frac{5}{3}}\sqrt{1+x^{\frac{4}{3}}}\,dx$$

To find this integral, use the substitution $u = 1 + x^{\frac{4}{3}}$. This gives $du = \frac{4}{3}x^{\frac{1}{3}}\,dx$, so

$$A = \tfrac{6\pi}{5}\int_0^1 x^{\frac{1}{3}}\,x^{\frac{4}{3}}\sqrt{1+x^{\frac{4}{3}}}\,dx = \tfrac{6\pi}{5}\int_1^2 \tfrac{3}{4}\,(u-1)\sqrt{u}\,du$$
$$= \tfrac{9\pi}{10}\int_1^2 (u^{\frac{3}{2}} - u^{\frac{1}{2}})\,du = \tfrac{9\pi}{10}\left[\tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}}\right]_1^2 = \tfrac{9\pi}{10}(\tfrac{2}{5}4\sqrt{2} - \tfrac{2}{3}2\sqrt{2} - \tfrac{2}{5} + \tfrac{2}{3}) = \tfrac{6\pi}{25}(\sqrt{2}+1)$$

Alternative solution: The substitution $u = x^{\frac{4}{3}}$ will also work, but requires integration by parts

[4] 11. Let $I = \int \sqrt{x}\,\ln(2x)\,dx$. Use integration by parts with $u = \ln(2x)$ and $dv = \sqrt{x}\,dx$

This choice gives $du = \frac{1}{x}\,dx$ and $v = \frac{2}{3}x^{\frac{3}{2}}$. We find

$$I = \tfrac{2}{3}x^{\frac{3}{2}}\ln(2x) - \int \tfrac{2}{3}x^{\frac{3}{2}}\cdot\tfrac{1}{x}\,dx$$
$$= \tfrac{2}{3}x^{\frac{3}{2}}\ln(2x) - \tfrac{2}{3}\int x^{\frac{1}{2}}\,dx$$
$$= \tfrac{2}{3}x^{\frac{3}{2}}\ln(2x) - \tfrac{4}{9}x^{\frac{3}{2}} + C$$

[4] 12. Divide all terms by the largest term in the denominator

$$\lim_{x\to\infty}\frac{x}{\sqrt{4x^2+9}} = \lim_{x\to\infty}\frac{\frac{x}{x}}{\sqrt{\frac{4x^2}{x^2}+\frac{9}{x^2}}} = \lim_{x\to\infty}\frac{1}{\sqrt{4+\frac{9}{x^2}}} = \frac{1}{\sqrt{4+0}} = \frac{1}{2}$$

Note: l'Hôpital's Rule will not work here. No score for trying to apply it